Math 275D Lecture 13 Notes

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1 Skorokhod's Representation Theorem

1.1 Skorokhod's representation theorem

We will study two new ways of understanding Brownian motion:

1. Brownian motion as a limit of simple random walks
2. Orthogonal polynomial method ot construct Brownian motion

The first one lets us prove results about Brownian motion using combinatorial arguments. The OPM is useful for computer simulation of Brownian motion and other applications.

Theorem 1.1 (Skorokhod's representation theorem)**.** *Let X be a real-valued random variable with $\mathbb{E}[X] = 0$. There exists a family of stopping times T_α with respect to $B(t)$ (where α is a random label) such that $B(T_\alpha) \stackrel{d}{=} X$ and $\mathbb{E}[T_\alpha] = \mathbb{E}[X^2]$.*

Example 1.1. Let $X = \pm 1$ with probability $1/2$ each. Let $T = \inf\{t : |B(t)| \geq 1\}$. Then $B(T) \sim X$ and $\mathbb{E}[T] = \mathbb{E}[X^2]$.

Example 1.2. Let $X = \pm 2$ with probability $1/2$ each. Then we can take $T = \inf\{t : |B(t)| \geq 2\}$.

Example 1.3. Let $X = \pm 1, \pm 2$ with probability $1/4$ each. Let $T_k = \inf\{t : |B(t)| \geq k\}$. Let $\alpha = 1$ or 2 with probability $1/2$ each. Then $B(T_\alpha) \stackrel{d}{=} X$.

Here is the outline of the proof.

Proof. Step 1: If $\mathbb{P}(X = a \text{ or } b) = 1$, then let $T_{a,b} = \inf\{t : B(t) = a \text{ or } b\}$.

Step 2: We want a random variable $\alpha : \Omega \to \mathbb{R}^2$ with a distribution such that $B(T_\alpha) \stackrel{d}{=} X$ and $\mathbb{E}[T_\alpha] = \mathbb{E}[X^2]$. In the discrete case, we have

$$\mathbb{P}(B(T_\alpha) = u) = \mathbb{E}_\alpha[\mathbb{P}_{\mathrm{BM}}(B(T_{u,v}) = u)] = \mathbb{E}_\alpha\left[\frac{v}{|u-v|}\right].$$

□

1.2 Proof of CLT using Skorohod's representation theorem

If we have the SLLN and this representation theorem, we can actually produce a proof of the central limit theorem.

Corollary 1.1 (CLT)**.** *Suppose that X_n are iid random variables with $\mathbb{E}[X_i] = 0$. Then*

$$\frac{\sum_{n=1}^{N} X_n}{\left(\sum_{n=1}^{N} X_n^2\right)^{1/2}} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. By Skorokhod's representation theorem, $X_1 \overset{d}{=} B(T_\alpha)$ and $X_2 \overset{d}{=} \tilde{B}(T_{\tilde{\alpha}})$, where $\alpha, \tilde{\alpha}$ are iid and $B \perp \tilde{B}$. Then $X_1 + X_2 \overset{d}{=} B(T_\alpha + \tilde{T}_{\tilde{\alpha}})$, where $\tilde{T}_{\tilde{\alpha}} = \inf\{t - T_\alpha : t > T_\alpha, B(t) - B(T_\alpha) \in \tilde{\alpha}\}$. (Recall $T_\alpha = \inf\{t > 0 : B(t) \in \alpha\}$.) The reason we can do this is that $B(T_\alpha + \tilde{T}_{\tilde{\alpha}}) - B(T_\alpha) \overset{d}{=} B(T_{\tilde{\alpha}})$. In fact, $T_\alpha, \tilde{T}_{\tilde{\alpha}}$ are iid.

We can extend this to $X_1 + X_2 + \cdots + X_n \overset{d}{=} B(T_{\alpha_1}^1 + T_{\alpha_2}^2 + \cdots + T_{\alpha_n}^n)$. By the SLLN, $\sum_n T_{\alpha_n}^n \to N\,\mathbb{E}[T_{\alpha_1}^1] = N\,\mathbb{E}[X^2]$. So we have $X_1 + X_n \overset{d}{=} B(Y_N)$, where $Y_n \to N\,\mathbb{E}[X_1^2]$ a.s. So

$$\frac{X_1 + \cdots + X_N}{\sqrt{N}} \xrightarrow{d} \mathcal{N}(0, \mathbb{E}[X^2]).$$

□

1.3 Brownian motion as a limit of simple random walks

Let $X_i \sim$ iid Ber(1/2), and let

$$S_X^N = \begin{cases} \sum_{k=1}^{m} X_k & \text{if } X = m, x \in \mathbb{N} \\ \text{linear combination of } S_{[x]}^n, S_{[x]+1}^n & x \notin \mathbb{N}. \end{cases}$$

In other words, we linearly interpolate between the values of a random walk. This gives us a graph (i.e. a random continuous function $\mathbb{R}_+ \to \mathbb{R}$). Then let

$$f^N(t) = \frac{S_{tN}^N}{\sqrt{N}}.$$

Then f^N converges in distribution to Brownian motion on $[0,1]$.

Usually convergence in distribution is not so strong. Next time, we will talk about how to improve this for our Brownian motion.